Ergodicity and indistinguishability in percolation

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In rough terms...

In 1999, Lyons and Schramm have proved in [LS] that when a Bernoulli percolation produces infinitely many infinite clusters, these clusters cannot be distinguished the one from the other. This indistinguishability result interacts with orbit equivalence theory as depicted below.

Percolation		Orbit equivalence
[LS] indistinguishability	$\underset{[G]}{\Longleftrightarrow}$	ergodicity
		$\Downarrow$ [CI]
?	$\Longleftrightarrow$	strong ergodicity

The aim of this poster is to explain this diagram and replace the question mark with a suitable notion of strong indistinguishability.

The indistinguishability theorem [LS]

ROUGH STATEMENT. *For Bernoulli percolation, no property can distinguish an infinite cluster from another.*

This statement is far from precise. In particular, one needs to define the word *property*. Before doing so, let us recall our general framework.

- **Graph** : $\mathcal{G} = (V, E)$ is a Cayley graph of a finitely generated group Γ. We shall denote by ρ some vertex in V that shall be referred to as the root.
- **Percolation** : Γ-invariant probability measures on $\Omega := 2^E$. Example : Bernoulli percolation $\mathbb{P}_p := \mathbf{Ber}(p)^{\otimes E}$.

What is meant by *property* ? [LS]

DEFINITION. A *vertex-property* is a Γ-invariant Borel Boolean function defined on $V \times \Omega$.

DEFINITION. A *cluster-property* is a vertex-property P that is "constant on clusters", that is whenever u and v are ω-connected,

$$P(u, \omega) = P(v, \omega)$$

Counter-examples :

- "u is the root" is not a vertex-property, since it is not Γ-invariant.
- "u is adjacent to four ω-open edges" is a vertex-property, but not a cluster-property.
- "The cluster of u contains the root" is not a vertex- or cluster-property, owing to the lack of Γ-invariance.

Examples of cluster-properties:

- "the ω-cluster of u has five ends"
- "the ω-cluster of u is transient"
- "the ω-cluster of u is at distance 1 from another infinite ω-cluster"

Back to the indistinguishability theorem [LS]

DEFINITION. A percolation is said to satisfy *indistinguishability* if for every cluster-property P, almost surely, all vertices that are in an infinite ω-cluster agree on P.

PRECISE STATEMENT. *Bernoulli percolation satisfies indistinguishability.*

This holds more generally for insertion-tolerant percolations.

Ergodicity and indistinguishability [G]

DEFINITION. The *cluster equivalence relation* R_{cl} is a countable Borel equivalence relation on Ω defined as follows:

$$\omega R_{cl} \omega' \Leftrightarrow \exists \gamma \in \Gamma, \rho \underset{\omega}{\longleftrightarrow} \gamma \cdot \rho \text{ and } \omega' = \gamma^{-1} \cdot \omega$$

One can think of an R_{cl}-class as a percolation configuration considered up to Γ-translation and together with the data of a "root-cluster".

PROPOSITION. *Let $\mathbb{P}$ denote a Γ-invariant percolation. Assume that $\Gamma \curvearrowright (\Omega, \mathbb{P})$ is ergodic. Then, $\mathbb{P}$ satisfies indistinguishability iff the relation R_{cl} restricted to its infinite locus is ergodic relative to $\mathbb{P}$.*

Subrelations of the Bernoulli shift [CI]

DEFINITION. We say that (X, R, μ) is *strongly ergodic* if, for every (R, μ)-asymptotically invariant sequence (B_n) of Borel subsets, the sequence $\mu(B_n)$ admits no other accumulation point than 0 and 1.

Recall that a sequence of Borel subsets is said to be (R, μ)-*asymptotically invariant* if, for any $\varphi \in [R]$, $\mu(\varphi(B_n) \triangle B_n)$ goes to 0.

THEOREM. *Let R be a subrelation of the Bernoulli shift $\Gamma \curvearrowright (\Omega, \mathbb{P}_p)$. If R, restricted to some Borel subset of Ω, is ergodic, then this restriction is strongly ergodic.*

Strong indistinguishability [M]

DEFINITION. Let $\mathbb{P}$ denote a Γ-invariant percolation. An *asymptotic cluster-property* is a sequence of vertex-properties P_n such that

$$\forall u, v \in V, \mathbb{P}[P(u, \omega) = P(v, \omega) | u \underset{\omega}{\longleftrightarrow} v] \underset{n \to \infty}{\longrightarrow} 1$$

DEFINITION. A percolation is said to satisfy *strong indistinguishability* if for every asymptotic cluster-property (P_n) and every $(u, v) \in V^2$,

$$\mathbb{P}[P(u, \omega) = P(v, \omega) | u \underset{\omega}{\longleftrightarrow} \infty \text{ and } v \underset{\omega}{\longleftrightarrow} \infty] \underset{n \to \infty}{\longrightarrow} 1$$

THEOREM. *Bernoulli percolation satisfies strong indistinguishability. Besides, in general, indistinguishability does not imply strong indistinguishability.*

[CI] I. CHIFAN and A. IOANA, *Ergodic subequivalence relations induced by a Bernoulli action*, Geom. Funct. Anal., vol. 20, p. 53-67, 2010.

[G] D. GABORIAU, *Invariant Percolation and Harmonic Dirichlet Functions*, Geom. Funct. Anal., vol. 15 (5), p. 1004-1051, 2005.

[LS] R. LYONS and O. SCHRAMM, *Indistinguishability of percolation clusters*, Ann. Probability, vol. 27 (4), p. 1809-1836, 1999.

[M] S. M. , *A note on ergodicity and indistinguishability*, to be submitted.

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